REMARKS ON SUBMULTIPLICATIVE FUNCTIONS

ABSTRACT. The real functions satisfying the inequality \( \Phi(uv) \leq K \Phi(u) \Phi(v) \) for some positive \( K \) which occur among others in [5], [3], [4], and referred there as submultiplicative, are discussed. A simplifying remark that \( \Phi \) satisfies this inequality iff \( K\Phi \) is submultiplicative in the standard sense, is done. It is shown that, under general conditions, the standard submultiplicativity of \( \Phi \) and the inequality \( \Phi(u) \Phi(\frac{1}{u}) \leq 1 \) imply that \( \Phi \) must be multiplicative. Applying a result of Bhatt [1], we observe that if \( p \) is a nontrivial seminorm on a Banach algebra \( X \) such that the set \( \{ p(x^2) : x \in X, p(x) \neq 0 \} \) is a singleton \( \{ \lambda \} \), then \( s = \lambda p \) is a submultiplicative seminorm on \( X \).

KEY WORDS: submultiplicative function, seminorm, Orlicz function, square property.

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1. Introduction

Submultiplicative functions, similarly as subadditive ones, frequently appear in applications, and have well-developed theories (see, for instance, Hille and Phillips, [2], Kuczma [6]). In some parts of functional analysis, especially concerned the Orlicz spaces, a nonstandard form of submultiplicativity occurs. In Krasnoselskij and Rutickij [5], (see also Hudzik and Maligranda, Mastyło, Persson [3], [4]) a function \( \Phi : [0, \infty) \to \mathbb{R} \) is referred to as submultiplicative on \([0, \infty)\), if there exists a positive constant \( K \) such that

\[
\Phi(uv) \leq K \Phi(u) \Phi(v) \quad \text{for all } u, v \geq 0.
\]

As for \( K = 1 \) we get the classical submultiplicativity, one could treat it as a generalization and, for convenience, we call it the submultiplicativity in the sense of Krasnoselskij and Rutickij.

We observe that \( \Phi \) is submultiplicative in this sense iff the function \( K\Phi \) is submultiplicative (see Theorem 1 in section 2). This fact allows to simplify notations and avoid introducing new notions of submultiplicativity. Moreover, in section 2 devoted to the standard submultiplicative functions,
we prove that if $\Phi : (0, \infty) \to (0, \infty)$ is submultiplicative on $(0, \infty)$ and $\Phi(u) \leq \frac{1}{\Phi(\frac{1}{u})}$ for all $u \in [1, \infty)$, then $\Phi$ is multiplicative on $(0, \infty)$.

In section 4, applying theorem of Bhatt [1] and Theorem 1, we conclude that if $p$ is a nontrivial seminorm on a Banach algebra $X$ such that the set $\{ \frac{p(x^2)}{p(x)} : x \in X, p(x) \neq 0 \}$ is a singleton $\{\lambda\}$, then $s = \lambda p$ is a submultiplicative seminorm on $X$.

2. Remark on classical submultiplicativity

A real valued function $\Phi$ defined on a set $C$ that is closed under multiplication, is called multiplicative on $C$, if

$$\Phi(uv) = \Phi(u) \Phi(v) \quad \text{for all } u, v \in C;$$

submultiplicative on $C$, if

$$\Phi(uv) \leq \Phi(u) \Phi(v) \quad \text{for all } u, v \in C,$$

and supermultiplicative on $C$, if the reversed inequality holds.

In the case of submultiplicative functions, simple considerations show that, without any loss of generality, one can assume that $O \notin C$ and the range of $\Phi$ is contained in the set of positive numbers.

Therefore, in this section, we assume the following

**Definition 1.** Let $I \subset (0, \infty)$ be an interval that is closed under multiplication. A function $\Phi : (0, \infty) \to (0, \infty)$ is called:

(i) multiplicative on $I$, if

$$\Phi(uv) = \Phi(u) \Phi(v), \quad u, v \in I;$$

(ii) submultiplicative on $I$, if

$$\Phi(uv) \leq \Phi(u) \Phi(v) \quad u, v \in I,$$

(iii) supermultiplicative on $I$, if the reversed inequality holds.

Setting $u = v = 1$, respectively, in (1) and (2) leads to

**Remark 1.** Let $\Phi : (0, \infty) \to (0, \infty)$.

(i) If $\Phi$ is multiplicative on $(0, \infty)$ then $\Phi(1) = 1$.

(ii) If $\Phi$ is submultiplicative on $(0, \infty)$ $\Phi(1) \geq 1$.

Let us note the following
Proposition 1. If a function \( \Phi : (0, \infty) \rightarrow (0, \infty) \) is submultiplicative on \((0, \infty)\) and
\[
\Phi(u) \leq \frac{1}{\Phi\left(\frac{1}{u}\right)}, \quad u \in [1, \infty),
\]
then \( \Phi \) is multiplicative on \((0, \infty)\).

Proof. From Remark 1 we have \(1 \leq \Phi(1)\). Hence, the submultiplicativity of \( \Phi \) implies that for all \( u > 0 \),
\[
1 \leq \Phi\left(\frac{1}{u}\right) \leq \Phi\left(\frac{1}{u}\right) \Phi(u),
\]
whence
\[
\Phi(u) \geq \frac{1}{\Phi\left(\frac{1}{u}\right)} u \geq 1.
\]
This inequality and (3) imply that, for all \( u \geq 1 \)
\[
\Phi(u) = \frac{1}{\Phi\left(\frac{1}{u}\right)} u \in [1, \infty),
\]
whence, obviously,
\[
\Phi(u) = \frac{1}{\Phi\left(\frac{1}{u}\right)} u \in (0, \infty).
\]

Applying in turn: the submultiplicativity of \( \Phi \); twice (4); the submultiplicativity of \( \Phi \); and again (4), we get, for all \( u, v \in (0, \infty) \),
\[
\Phi(uv) \leq \Phi(u) \Phi(v) = \frac{1}{\Phi\left(\frac{1}{u}\right)} \Phi\left(\frac{1}{v}\right) \leq \frac{1}{\Phi\left(\frac{1}{uv}\right)} = \Phi(uv).
\]

\[\blacksquare\]

Remark 2. If \( \Phi : (0, \infty) \rightarrow (0, \infty) \) multiplicative the graph of \( \Phi \) is not dense in \((0, \infty)^2\) or \( \Phi \) is Lebesgue measurable, then there is \( p \in \mathbb{R} \) such that
\[
\Phi(u) = u^p, \quad u \in (0, \infty).
\]

The theory of subadditive function (cf. Hille-Phillips [2], Kuczma [6]) leads to the following

Remark 3. If \( \Phi : (0, \infty) \rightarrow (0, \infty) \) is submultiplicative continuous at 1 and \( \Phi(1) \leq 1 \) then \( \Phi \) is continuous.

Similarly, making use of the main result of [7] one gets the following
Remark 4. If $\Phi : (1, \infty) \to (1, \infty)$ is one-to-one, submultiplicative on $(1, \infty)$ and $\lim_{u \to 1^+} \Phi (u) = 1$, then $\Phi$ is continuous.

Example 1. Let $p, q \in (0, \infty), 0 < q \leq 1 \leq p$ be arbitrarily fixed. Then the function $\Phi : (0, \infty) \to (0, \infty)$ defined by
\[
\Phi (u) := \begin{cases} 
    u^q & \text{if } u \in (0, 1) \\
    u^p & \text{if } u \in [1, \infty)
\end{cases}
\]
is submultiplicative on $(0, \infty)$.

3. Submultiplicativity in the sense of Krasnoselskij and Rutickij

In Krasnoselskij and Rutickij [5], Hudzik and Maligranda [3]), a function $\Phi : [0, \infty) \to \mathbb{R}$ is referred to as submultiplicative on $[0, \infty)$, if it satisfies the following condition:

There exists a positive constant $K$ such that
\[
\Phi (uv) \leq K \Phi (u) \Phi (v) \quad \text{for all } u, v \geq 0.
\]

Let us note the following obvious

Remark 5. Every nonpositive function $\Phi : [0, \infty) \to \mathbb{R}$ satisfies inequality (5) with arbitrary $K \geq 0$.

Remark 6. Let $\Phi : [0, \infty) \to \mathbb{R}$ be an arbitrary function satisfying (5) with some $K \geq 0$.

If $\Phi (u_0) = 0$ for some $u_0 > 0$ then $\Phi (u) \leq 0$ for all $u \geq 0$.

Proof. For every $u \geq 0$, making use of (5), we have
\[
\Phi (u) = \Phi \left( u_0 \frac{u}{u_0} \right) \leq K \Phi (u_0) \Phi (u_0) = 0.
\]

Replacing in (5): ”$u, v \geq 0$” by ”$u, v > 0$” we obtain a weaker condition than (5). Moreover, as the interval $(0, \infty)$ is a multiplicative group, the set of all positive real numbers seems to be more convenient in examination of submultiplicativity than $[0, \infty)$.

Taking into account the above remarks, one can propose the following

Definition 2. A function $\Phi : (0, \infty) \to [0, \infty)$ is submultiplicative in the Krasnoselskij-Rutickij sense on $(0, \infty)$, if there exists a positive constant $K$ such that
\[
\Phi (uv) \leq K \Phi (u) \Phi (v), \quad u, v \in (0, \infty).
\]
Theorem 1. Let $\Phi : (0, \infty) \to [0, \infty)$. Then

(i) $\Phi$ is submultiplicative in the Krasnoselskij-Rutickij sense on $(0, \infty)$ if, and only if, for some positive real $K$, the function $K\Phi$ is submultiplicative in the classical sense (Definition 1, (ii));

(ii) if $\Phi$ is submultiplicative, then for every $K \geq 1$, the function $K\Phi$ is submultiplicative;

(iii) if $K \in (0, 1]$ and $K\Phi$ is submultiplicative, then $\Phi$ is submultiplicative;

(iv) if $K > 1$ and $K\Phi$ is submultiplicative, then $\Phi$ need not be submultiplicative.

Proof. To show (i) note that inequality (6) is equivalent to the inequality

$$K\Phi(uv) \leq [K\Phi(u)][K\Phi(v)]$$

for all $u, v \geq 0$, that is equivalent to the sumultiplicativity of the function $K\Phi$.

(ii) and (iii) are easy to verify.

To prove (iv) take arbitrary $K > 1$ and $p \in \mathbb{R}$, an consider the function $\Phi : (0, \infty) \to (0, \infty)$, $\Phi(u) := \frac{1}{K} u^p$. Of course the function $K\Phi(u) = u^p$, being multiplicative, is submultiplicative. Since the inequality (6) holds iff

$$\frac{1}{K} (uv)^p \leq \frac{1}{K^2} u^p v^p$$

for all $u, v > 0$, that is iff $K \leq 1$, the function $\Phi$ is not submultiplicative.

Remark 7. Hudzik and Maligranda [3] gave a negative answer to the question posed in [5], p. 301, whether or not for any Orlicz function $\Phi$ which is submultiplicative at infinity in the Krasnoselskij-Rutickij sense (i.e. such that $\Phi(uv) \leq K\Phi(u)\Phi(v)$ for all $u, v \geq u_0$ for some positive $K$ and nonnegative $u_0$) there exists an Orlicz function $\Psi$ which is equivalent to $\Phi$ at infinity, submultiplicative on $[0, \infty)$, and such that

$$\lim_{u \to 0} \frac{\Psi(u)}{u} = 0.$$

4. Remark on submultiplicative seminorms on Banach algebra

Let $X$ be an algebra over the real or complex numbers $\mathbb{K}$. A seminorm on $X$ is a function $s : X \to [0, \infty)$ such that it is homogeneous and subadditive, i.e.

$$s(tx) = |t|s(x), \quad s(x + y) \leq s(x) + s(y)$$

for all $t \in \mathbb{K}$ and $x, y \in X$. A seminorm $s$ is called submultiplicative, if

$$s(xy) \leq s(x)s(y), \quad x, y \in X.$$

We prove the following
Theorem 2. Let $X$ be an algebra over $\mathbb{K}$. If $p : X \rightarrow \mathbb{R}$ is a function such that

\begin{align*}
\text{(7)} & \quad p(tx) \leq |t|p(x), \quad x \in X, \ t \in \mathbb{K}; \\
\text{(8)} & \quad p(x + y) \leq p(x) + p(y), \quad x, y \in X,
\end{align*}

and there is a positive constant $\lambda$ such that

\begin{equation}
\text{(9)} \quad p(xy) \leq \lambda p(x)p(y), \quad x, y \in X,
\end{equation}

then $s := \lambda p$ is a submultiplicative seminorm on $X$.

Proof. Take arbitrary $t \in \mathbb{K}$, $t \neq 0$ and $x \in X$. Replacing $t$ by $\frac{1}{t}$ and $x$ by $tx$ in inequality (7) we get $|t|p(x) \leq p(tx)$ so, taking into account (7), we get

\begin{equation}
\text{(10)} \quad p(tx) = |t|p(x)
\end{equation}

for all $x \in X$, $t \in \mathbb{K}$, $t \neq 0$.

From (7), for $t = 0$ and $x \in X$ we have $p(0x) = p(0) \leq 0$. On the other hand, from (8) with $x = y = 0$, we get $0 \leq p(0)$. So, equality (10) holds for all $x \in X$, $t \in \mathbb{K}$, which proves that $p$ is homogeneous.

Hence, applying in turn: subadditivity of $p$ an homogeneity, we get, for all $x \in X$,

\[ 0 = p(0) = p(x + (-x)) \leq p(x) + p(-x) = 2p(x), \]

which shows that $p : X \rightarrow [0, \infty)$, that is $p$ is nonnegative.

Since $\lambda$ is positive, clearly, the function $s := \lambda p$ is nonnegative, homogeneous and subadditive, so $s$ is a seminorm on $X$. Moreover, multiplying both sides of inequality (9) by $\lambda$, we get

\[ (\lambda p)(xy) \leq (\lambda p)(x)(\lambda p)(y), \quad x, y \in X, \]

that is

\[ s(xy) \leq s(x)s(y), \quad x, y \in X, \]

which shows that $s$ is submultiplicative. ■

Remark 8. Let $p : X \rightarrow \mathbb{K}$ be a nonzero seminorm on a Banach algebra $X$ such that the set

\[ \left\{ \frac{p(x^2)}{p(x)^2} : x \in X, \ p(x) \neq 0 \right\} \]

is a singleton $\{\lambda\}$. Then $s = \lambda p$ is a submultiplicative seminorm on $X$. 


Remarks on submultiplicative functions

Proof. By Theorem 1 the function \( s := \lambda p \) is seminorm. By the definition of \( \lambda \) we have, for all \( x \in X \),

\[
(x^2) = \lambda p (x^2) = \lambda (\lambda [p(x)]^2) = (\lambda [p(x)]) (\lambda [p(x)]) = [s (x)]^2,
\]

which shows that \( s \) has the so-called square property. In view of theorem of Bhatt [1], the seminorm \( s \) is submultiplicative. ■

References


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