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A FUGLEDE–PUTNAM TYPE THEOREM FOR A CLASS OF ALMOST NORMAL OPERATORS

Abstract. In this note we will prove that operators $T \in \mathcal{L}(\mathcal{H})$ with finite k_1 function satisfy a Fuglede−Putnam type modulo the Hilbert−Schmidt class, that is, for arbitrary $X \in \mathcal{L}(\mathcal{H})$ with $TX - XT \in \mathcal{C}_2(\mathcal{H})$ implies $T^*X - XT^* \in \mathcal{C}_2(\mathcal{H})$.

Key words: almost normal operators, k_1-function, Fuglede −Putnam type theorem.

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1. Introduction

1. Let $\mathcal{H}$ be a separable, infinite dimensional, complex Hilbert space, and denote by $\mathcal{L}(\mathcal{H})$ the algebra of all bounded linear operators on $\mathcal{H}$ and by $\mathcal{C}_p(\mathcal{H})$ (or simply $\mathcal{C}_p$) the Shatten−von Neumann p-classes and by $|\cdot|_p$, $p \geq 1$, their respective norm. In this note, of particular interest will be the classes corresponding to $p = 1, 2$, that is the trace-class $\mathcal{C}_1$ and the class of Hilbert−Schmidt operators $\mathcal{C}_2$. For arbitrary operators $S, T \in \mathcal{L}(\mathcal{H})$, $[S, T]$ will denote their commutator $ST - TS$ and D_S will denote the self-commutator of S, that is $[S^*, S]$. An operator $S \in \mathcal{L}(\mathcal{H})$ is called *almost normal* when $D_S \in \mathcal{C}_1(\mathcal{H})$ and the class of operators defined on $\mathcal{H}$ which are almost normal will be denoted by $\mathcal{AN}(\mathcal{H})$.

2. Voiculescu's Conjecture 4 (C_4) (cf. [3] or [4]) states that for $T \in \mathcal{AN}(\mathcal{H})$, there exists $S \in \mathcal{AN}(\mathcal{H})$ such that $T \oplus S = N + K$, where N is a normal operator and K is a Hilbert−Schmidt operator. Under the assumption that conjecture (C_4) has a positive answer, one can easily prove that almost normal operators satisfy a Fuglede−Putnam type theorem, that is, if $T \in \mathcal{AN}(\mathcal{H})$ and $X \in \mathcal{L}(\mathcal{H})$ is such that $[T, X] \in \mathcal{C}_2(\mathcal{H})$, then $[T^*, X] \in \mathcal{C}_2(\mathcal{H})$, and the family of operators on $\mathcal{H}$ that have such a property will be denoted by $\mathcal{FP}_2(\mathcal{H})$. This is an easy consequence of a theorem of G. Weiss [5] that states that if $N \in \mathcal{L}(\mathcal{H})$ is a normal operator and $X \in \mathcal{L}(\mathcal{H})$ such that $[N, X] \in \mathcal{C}_2(\mathcal{H})$, then $[N^*, X] \in \mathcal{C}_2(\mathcal{H})$ and $|[N, X]|_2 = |[N^*, X]|_2$ (in particular $N \in \mathcal{FP}_2(\mathcal{H})$), and the details are left for the reader.

Let $\mathcal{P}$ and $\mathcal{R}_1^+$ denote the set of finite rank orthogonal projections and the finite rank positive semidefinite contractions respectively, and

$$\begin{aligned} q_p(T) &= \liminf_{P\in\mathcal{P}} |((I-P)TP)|_p, \\ k_p(T) &= \liminf_{A\in\mathcal{R}_1^+} |[T,A]|_p, \end{aligned}$$

where the lim inf's are with respect to the natural order.

3. In [1] it was proved that almost normal operators T such that $q_2(T) < \infty$ belong to $\mathcal{FP}_2(\mathcal{H})$. It is natural to ask whether almost normal operators T with finite $k_2(T)$, and implicitly $k_2(T) = 0$ (since the function k_2 is either zero or infinite acc. [2]), belong to $\mathcal{FP}_2$.

In this note we will prove that such a result holds under the hypothesis that $k_1(T)$ is finite. We mention that k_1 is not necessarily zero when it is finite.

Theorem 1. *If $T \in \mathcal{AN}(\mathcal{H})$ and $k_1(T) < \infty$, then for $X \in \mathcal{L}(\mathcal{H})$ the commutator $[T,X]$ is a Hilbert–Schmidt operator if and only if so is $[T^*,X]$.*

Proof. Let $T \in \mathcal{AN}(\mathcal{H})$ with $k_1(T) < \infty$, let $A_n \in \mathcal{R}_1^+$, $n \geq 1$, so that $A_n \uparrow I$ and $|[A_n,T]|_1 \downarrow k_1(T)$, and let $X \in \mathcal{L}(\mathcal{H})$ with $[T,X] =: R \in \mathcal{C}_2(\mathcal{H})$.

It will be enough to prove that

$$\limsup_{n\to\infty} |\mathrm{tr}[A_n(QQ^* - RR^*)]| < \infty,$$

where $Q := T^*X - XT^*$. Write

$$\begin{aligned} A_nRR^* &= A_nTXX^*T^* - A_nTXT^*X^* - A_nXTX^*T^* + A_nXTT^*X^* \\ &= a - b - c + d \end{aligned}$$

and

$$\begin{aligned} A_nQQ^* &= A_nT^*XX^*T - A_nT^*XTX^* - A_nXT^*X^*T + A_nXT^*TX^* \\ &= A - B - C + D, \end{aligned}$$

where a, b, ..., C, D are the terms in the order they appear in these expansions. We will use several times each of the following facts about trace-class operators: $\mathrm{tr}([F,Y]) = 0$, $|\mathrm{tr}(F)| \leq |F|_1$, and $|FY|_1$, $|YF|_1 \leq |F|_1\,||Y||$, for $F \in \mathcal{C}_1$ and in particular for finite rank operators, and arbitrary $Y \in \mathcal{L}(\mathcal{H})$.

First

$$|\mathrm{tr}(D-d)| \leq |D-d|_1 \leq ||X||^2\,|D_T|_1. \tag{1}$$

Then

$$\begin{aligned}|\mathrm{tr}(B-c)| &= |\mathrm{tr}(A_nT^*XTX^* - A_nXTX^*T^*)|\\ &= |\mathrm{tr}(A_nT^*XTX^* - T^*A_nXTX^*)| = |\mathrm{tr}([A_n,T^*]XTX^*)|\\ &\le |[A_n,T^*]XTX^*|_1 \le |[A_n,T^*]|_1||XTX^*||\\ &\le |[A_n,T^*]|_1\,||X||^2\,||T||\end{aligned}$$

and then after passing to limit

$$|\mathrm{tr}(B-c)| \le k_1(T)\,||X||^2\,||T||. \tag{2}$$

In a similar way,

$$\begin{aligned}|\mathrm{tr}(C-b)| &= |\mathrm{tr}(A_nXT^*X^*T - A_nTXT^*X^*)|\\ &= |\mathrm{tr}(TA_nXT^*X^* - T^*A_nTXT^*X^*)| = |\mathrm{tr}([T,A_n]XT^*X^*)|\\ &\le |[A_n,T]XT^*X^*|_1 \le |[A_n,T]|_1||XT^*X^*||\\ &\le |[A_n,T]|_1\,||X||^2\,||T||\end{aligned}$$

and thus

$$|\mathrm{tr}(C-b)| \le k_1(T)\,||X||^2\,||T||. \tag{3}$$

Finally,

$$\begin{aligned}|\mathrm{tr}(A-a)| &= |\mathrm{tr}(A_nT^*XX^*T - A_nTXX^*T^*)|\\ &= |\mathrm{tr}(TA_nT^*XX^* - T^*A_nTXX^*)|\\ &\le |TA_nT^* - T^*A_nT|_1\,||X||^2.\end{aligned}$$

Furthermore

$$\begin{aligned}|TA_nT^* - T^*A_nT|_1 &= |(TA_nT^* - A_nTT^*)\\ &\quad + (A_nTT^* - A_nT^*T) + (A_nT^*T - T^*A_nT)|_1\\ &\le |[[T,A_n]T^*|_1 + |A_nD_T|_1 + |[A_n,T^*]T|_1\\ &\le |[T,A_n]|_1\,||T^*|| + |D_T|_1 + |[A_n,T^*]|_1\,||T||\\ &= 2\,|[T,A_n]|_1\,||T|| + |D_T|_1,\end{aligned}$$

and consequently, by passing to limit, we have

$$|\mathrm{tr}(A-a)| \le (2k_1(T)\,||T|| + |D_T|_1)\,||X||^2. \tag{4}$$

Using inequalities (1)-(4),

$$\limsup_{n\to\infty} |\mathrm{tr}[A_n(QQ^* - RR^*)]| \le 4k_1(T)||X||^2||T|| + 2|D_T|_1||X||^2,$$

which proves one implication of the theorem. The other implication is a consequence of the previous one since $k_1(T) = k_1(T^*)$. ■

The above proof leads to the following.

Corollary 1. *If* $T \in \mathcal{AN}(\mathcal{H})$ *with* $k_1(T) < \infty$ *and* $X \in \mathcal{L}(\mathcal{H})$ *so that* $[T, X] \in \mathcal{C}_2(\mathcal{H})$*, then* $|[T^*, X]|_2^2 \leq |[T, X]|_2^2 + 4k_1(T)||X||^2||T|| + 2|D_T|_1||X||^2$.

Corollary 2. *If* $T, S \in \mathcal{AN}(\mathcal{H})$ *with* $k_1(T)$ *and* $k_1(S) < \infty$ *and* $X \in \mathcal{L}(\mathcal{H})$ *so that* $R := TX - XS \in \mathcal{C}_2(\mathcal{H})$*, then* $Q := T^*X - XS^* \in \mathcal{C}_2(\mathcal{H})$ *and*

$$\begin{aligned}|Q|_2^2 \ \leq \ & |R|_2^2 + 4(k_1(T) + k_1(S))||X||^2 \max\{||T||, ||S||\} \\ & + 2(|D_T|_1 + |D_S|_1)||X||^2.\end{aligned}$$

Proof. Let T, S, X as in the hypothesis. It is straightforward to see that

$$k_1(T \oplus S) \leq k_1(T) + k_1(S) < \infty.$$

Setting $\tilde{X} = \begin{pmatrix} 0 & X \\ 0 & 0 \end{pmatrix}$ then $(T \oplus S)\tilde{X} - \tilde{X}(T \oplus S) = \begin{pmatrix} 0 & R \\ 0 & 0 \end{pmatrix}$, and thus $(T \oplus S)\tilde{X} - \tilde{X}(T \oplus S) \in \mathcal{C}_2(\mathcal{H})$. Therefore $(T \oplus S)^*\tilde{X} - \tilde{X}(T \oplus S)^* = \begin{pmatrix} 0 & R \\ 0 & 0 \end{pmatrix} \in \mathcal{C}_2$. Consequently $Q \in \mathcal{C}_2(\mathcal{H})$, and the inequality is left for the reader. ■

Corollary 3. *If* $T \in \mathcal{AN}(\mathcal{H})$ *with* $k_1(T) < \infty$ *and* $X \in \mathcal{L}(\mathcal{H})$ *so that* $R' := TX - XT^* \in \mathcal{C}_2(\mathcal{H})$*, then* $Q' := T^*X - XT \in \mathcal{C}_2(\mathcal{H})$ *and*

$$|Q'|_2^2 \leq |R'|_2^2 + 8k_1(T)||X||^2||T|| + 4|D_T|_1||X||^2.$$

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